

On the Computation of π -Flat Outputs for Linear Time-Delay Systems[☆]

F. Cazaurang

*University of Bordeaux, IMS Laboratory, CNRS, UMR 5218, 351 cours de la Libération,
33405 Bordeaux, France.*

J. Lévine

*Centre Automatique et Systèmes (CAS), Unité Mathématiques et Systèmes,
MINES-ParisTech, 35 rue Saint-Honoré, 77300 Fontainebleau, France.*

V. Morio^{☆☆}

*University of Bordeaux, IMS Laboratory, CNRS, UMR 5218, 351 cours de la Libération,
33405 Bordeaux, France.*

Abstract

This paper deals with the concepts of π -flatness and π -flat output for linear time-varying delay systems. These notions, introduced and developed by several authors during the last decade, may be, roughly speaking, defined as follows: a π -flat system is a system for which all its variables may be expressed as functions of a particular output y , a finite number of its successive time derivatives, time delays, and predictions, the latter resulting from the advance operator π^{-1} , π being a polynomial of δ , the delay operator. Thanks to standard polynomial algebraic tools, and in particular the Smith-Jacobson decomposition of polynomial matrices, we obtain a simple and easily computable characterization of π -flatness in terms of *hyper-regularity* of the system matrices and deduce a constructive algorithm for the computation of π -flat outputs. Some examples are provided to illustrate the proposed methodology.

Key words: linear systems, time-varying systems, delay systems, polynomial matrices, matrix decomposition, π -flatness, π -flat output.

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^{☆☆}Now with DGA/DT/LRBA/SDT/MAN/PSP, BP 914, 27207 Vernon Cedex, France.
Email addresses: franck.cazaurang@ims-bordeaux.fr (F. Cazaurang),
jean.levine@mines-paristech.fr (J. Lévine), vincent.morio@dga.defense.gouv.fr
(V. Morio^{☆☆})

1. Introduction

Differential flatness, roughly speaking, means that all the variables of an under-determined system of differential equations can be expressed as functions of a particular output, called flat output, and a finite number of its successive time derivatives ([20, 8, 9], see also [33, 16, 17] and the references therein).

For time-delay systems and more general classes of infinite-dimensional systems, extensions of this concept have been proposed and thoroughly discussed in [22, 10, 24, 31]. In a linear context, relations with the notion of system parameterization [28, 27] and, in the behavioral approach of [26], with *latent variables* of *observable image representations* [35], have been established. Other theoretic approaches have been proposed e.g. in [30, 3]. Interesting control applications of linear time-delay systems may be found in [22, 24, 31].

Characterizing differential flatness and flat outputs has been an active topic since the beginning of this theory. The interested reader may find a historical perspective of this question in [16, 17]. Constructive algorithms, relying on standard computer algebra environments, may be found e.g. in [1] for nonlinear finite-dimensional systems, or [2] for linear systems over Ore algebras.

The results and algorithm proposed in this paper for the characterization and computation of π -flat outputs for linear time-delay systems are strongly related to the algebraic framework developed in [22, 25, 31]. More precisely, we study linear time-delay differential control systems, i.e. linear systems of the form $Ax = Bu$, with $x \in \mathbb{R}^n$ the pseudo-state, and $u \in \mathbb{R}^m$ the control, for given integers $m \leq n$, where the entries of the matrices A and B belong to the ring $\mathfrak{K}[\delta, \frac{d}{dt}]$ of multivariate polynomials of δ , the delay operator, and $\frac{d}{dt}$, the time derivative operator, over the ground field \mathfrak{K} of meromorphic functions of the variable t . We say that the system $Ax = Bu$ is π -flat if, and only if, the module generated by the components of x and u over $\mathfrak{K}[\delta, \frac{d}{dt}]$ and satisfying the relations $Ax = Bu$, localized at the powers of a polynomial $\pi \in \mathfrak{K}[\delta]$, is free, and a π -flat output is a basis of this free module (see [22]).

To characterize and compute π -flat outputs, we propose a methodology based on standard polynomial algebra, generalizing the one used in [18] for ordinary linear differential systems, by extending the original ring $\mathfrak{K}[\delta, \frac{d}{dt}]$ to the principal ideal ring $\mathfrak{K}(\delta)[\frac{d}{dt}]$ of polynomials of $\frac{d}{dt}$ over the fraction field $\mathfrak{K}(\delta)$, namely the field generated by fractions of polynomials of δ with coefficients in \mathfrak{K} , and finally localize the results of our computations at the powers of a suitable polynomial π of $\mathfrak{K}[\delta]$. This approach allows us to use the well-known Smith-Jacobson (or diagonal) decomposition ([5, 13]) of matrices with entries in the larger ring $\mathfrak{K}(\delta)[\frac{d}{dt}]$ as the main tool to obtain the searched π -flat outputs.

Following [18], in order to work with a smaller set of equations and variables, we eliminate the input variables, leading to an implicit system representation, as opposed to previous approaches (see e.g. [22, 24, 31, 3, 4]). Let us also insist on the fact that the time-varying dependence of the systems under consideration is in the class of meromorphic functions, whereas in [3, 4], this dependence is polynomial with respect to time in order to apply effective Gröbner bases techniques.

The main contributions of this paper are (1) the characterization of π -flatness in terms of the *hyper-regularity* of the system matrices, (2) yielding an elementary algorithm to compute π -flat outputs, based on the Smith-Jacobson decomposition of the former matrices. In addition, the evaluation of our π -flatness criterion only relies on computations over the larger ring $\mathfrak{K}(\delta)[\frac{d}{dt}]$.

The paper is organized as follows. The π -flat output computation problem is described in section 2, as well as the algebraic framework. Then, the main result of the paper is presented in section 3. Finally, the proposed methodology is illustrated by some examples in section 4, and its generalization to multiple delays is outlined on an example of vibrating string, first solved in [23].

2. Problem Statement

We consider a linear system governed by the set of time-delay differential equations:

$$A\left(\delta, \frac{d}{dt}\right)x = B\left(\delta, \frac{d}{dt}\right)u, \quad (1)$$

where $x \in \mathbb{R}^n$ is the pseudo-state, $u \in \mathbb{R}^m$ the input vector, A (resp. B) a $n \times n$ (resp. $n \times m$) matrix, whose coefficients are multivariate polynomials of δ and $\frac{d}{dt}$, with $\frac{d}{dt}$ the differentiation operator with respect to time and δ the time-delay operator defined by:

$$\delta : f(t) \mapsto \delta f(t) = f(t - \tau), \quad \forall t \in \mathbb{R} \quad (2)$$

where $\tau \in \mathbb{R}^+$ is the delay.

In order to precise the nature of the coefficients $a_{i,j}(\delta, \frac{d}{dt})$, $i, j = 1, \dots, n$, and $b_{i,j}(\delta, \frac{d}{dt})$, $i = 1, \dots, n$, $j = 1, \dots, m$, of the matrices $A(\delta, \frac{d}{dt})$ and $B(\delta, \frac{d}{dt})$ respectively, some algebraic recalls are needed.

2.1. Algebraic Framework

Since we deal with smooth functions of time, a natural field is the *differential field of meromorphic functions on the real line* \mathbb{R} . We call this field the *ground field* and we denote it by \mathfrak{K} . The previously introduced operators δ and $\frac{d}{dt}$ satisfy the following rules:

$$\frac{d}{dt}(\alpha(t) \cdot) = \alpha(t) \frac{d}{dt} + \dot{\alpha}(t) \cdot, \quad \delta(\alpha(t) \cdot) = \alpha(t - \tau) \delta, \quad \frac{d}{dt} \delta = \delta \frac{d}{dt}$$

for every time function α belonging to \mathfrak{K} . The set of multivariate polynomials of these operators, namely polynomials of the form

$$\sum_{k,l \text{ finite}} \alpha_{k,l}(t) \frac{d^k}{dt^k} \delta^l, \quad \alpha_{k,l} \in \mathfrak{K}$$

is a *skew commutative ring* [21, 32], denoted by $\mathfrak{K}[\delta, \frac{d}{dt}]$. The coefficients $a_{i,j}$ (resp. $b_{i,j}$) of the matrix A (resp. B) of system (1) are supposed to belong to $\mathfrak{K}[\delta, \frac{d}{dt}]$, thus making system (1) a linear time-varying time-delay differential system, whose coefficients are meromorphic functions with respect to time.

2.1.1. System Module, Freeness

To system (1) is associated the so-called *system module*, noted Λ . More precisely, following [7, 22], let us consider a non zero, but otherwise arbitrary, pair $(\xi, \nu) = (\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_m)$ and the free module¹, denoted by $[\xi, \nu]$, generated by all possible linear combinations of ξ and ν with coefficients in $\mathfrak{K}[\delta, \frac{d}{dt}]$. Next, we set $\theta = A\xi - B\nu$ and construct the submodule $[\theta]$ of $[\xi, \nu]$ generated by the components of the vector θ . The system module Λ is, by definition, the quotient module $\Lambda = [\xi, \nu]/[\theta]$.

In [22], in the context of commutative polynomial rings, the notion of projective (resp. torsion-free) *controllability* of a time-invariant system, i.e. a system of the form (1) with ground field $\mathfrak{K} = \mathbb{R}$, is defined as the projective (resp. torsion) freeness of Λ , and shown to generalize the well-known Kalman controllability criterion to linear time-invariant differential delay systems. Moreover, as a consequence of a theorem of Quillen and Suslin, solving a conjecture of Serre (see e.g. [6, 15]), Λ is free if and only if it is projective free. If F is a finite-dimensional presentation matrix of Λ , the latter module Λ is projective free if F is right-invertible, i.e. there exists a matrix T over $\mathfrak{K}[\delta, \frac{d}{dt}]$ such that $FT = I$.

This approach has been generalized to modules over the Weyl algebras by Quadrat and Robertz [29], based on a theorem of Stafford [34], (see algorithmic versions of this result in [12, 19]).

In both time-invariant and time-varying cases, systems whose module is free are called *flat* ([22, 25, 31]). Nevertheless, only few systems have a free system module, thus motivating the weaker notion of π -flatness: we say that the system is π -flat, or that its associated module is π -free ([22, 31]), if, and only if, there exists a polynomial $\pi \in \mathfrak{K}[\delta]$, called *liberation polynomial* ([22, 31]), such that the module $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}] \otimes_{\mathfrak{K}[\delta, \frac{d}{dt}]} \Lambda$, i.e. the set of elements of the form $\sum_{i \in I} \pi^{-i} a_i \xi_i$ with I arbitrary subset of \mathbb{N} , $a_i \in \mathfrak{K}[\delta, \frac{d}{dt}]$ and $\xi_i \in \Lambda$ for all $i \in I$, called the *system module localized at the powers of π* , is free. In other words, π -flatness means that the state and input can be expressed in terms of the π -flat output, a finite number of its time derivatives and delays, and advances corresponding to powers of the inverse operator π^{-1} .

In the sequel, we will also use the extension, as announced, of the ground field \mathfrak{K} to $\mathfrak{K}(\delta)$, the fraction field generated by rational functions of δ with coefficients in \mathfrak{K} . The system module over this field extension is $\mathfrak{K}(\delta)[\frac{d}{dt}] \otimes_{\mathfrak{K}[\delta, \frac{d}{dt}]} \Lambda$. Indeed, freeness (in any sense) of the latter module does not imply freeness (in any sense) of the original system module Λ (see e.g. [22]).

2.1.2. Polynomial Matrices, Smith-Jacobson Decomposition, Hyper-regularity

The matrices of size $p \times q$ whose entries are in $\mathfrak{K}[\delta, \frac{d}{dt}]$ generate a *module* denoted by $\mathcal{M}_{p,q}[\delta, \frac{d}{dt}]$. The matrix $M \in \mathcal{M}_{n,n}[\delta, \frac{d}{dt}]$ is said to be *invertible*² if

¹For more details on rings and modules, the reader may refer to [5].

²Note that the $\mathfrak{K}[\delta, \frac{d}{dt}]$ -independance of the n columns and rows of M is not sufficient for its invertibility. Its inverse, denoted by N , has to be polynomial too.

there exists a matrix $N \in \mathcal{M}_{n,n}[\delta, \frac{d}{dt}]$ such that $MN = NM = I_n$, where I_n is the identity matrix of order n . The subgroup of $\mathcal{M}_{n,n}[\delta, \frac{d}{dt}]$ of invertible matrices is called the group of *unimodular matrices* of size n and is noted $\mathcal{U}_n[\delta, \frac{d}{dt}]$ ³.

Let us give an example of a system of the form (1), that will serve as a guideline all along this section to illustrate the various concepts.

Example 1.

$$Ax \triangleq \begin{pmatrix} \frac{d}{dt} & -k(t)(\delta - \delta^2) \\ 0 & \frac{d}{dt} \end{pmatrix} x = \begin{pmatrix} 0 \\ \delta \end{pmatrix} u \triangleq Bu \quad (3)$$

where $x = (x_1, x_2)^T$, u is scalar and $k(t)$ a meromorphic function. In other words (3) reads:

$$\begin{cases} \dot{x}_1(t) = k(t)(x_2(t - \tau) - x_2(t - 2\tau)) \\ \dot{x}_2(t) = u(t - \tau) \end{cases} \quad (4)$$

The coefficients $\frac{d}{dt}$ and $-k(t)(\delta - \delta^2) = -k(t)\delta(1 - \delta)$ are elements of $\mathfrak{K}[\delta, \frac{d}{dt}]$ and the corresponding matrices A and B belong to $\mathcal{M}_{2,2}[\delta, \frac{d}{dt}]$ and $\mathcal{M}_{2,1}[\delta, \frac{d}{dt}]$ respectively.

Note that it may be necessary to extend the polynomial ring as shown by the following computation on the previous example:

Let us express u of (3) as a function of x_1 . It is straightforward to see that $x_2 = (1 - \delta)^{-1}\delta^{-1}\frac{1}{k}\frac{d}{dt}x_1$ and, since $u = \delta^{-1}\dot{x}_2$, we immediately get $u = \delta^{-2}(1 - \delta)^{-1}\left(-\frac{\dot{k}}{k^2} + \frac{1}{k}\frac{d}{dt}\right)\frac{d}{dt}x_1$. Denoting by $\pi \triangleq (1 - \delta)\delta^2 \in \mathfrak{K}[\delta]$, the polynomial $\pi^{-1}\left(-\frac{\dot{k}}{k^2} + \frac{1}{k}\frac{d}{dt}\right)\frac{d}{dt}$ lives in $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ the ring of polynomials of δ, π^{-1} and $\frac{d}{dt}$ with coefficients in \mathfrak{K} , but not in $\mathfrak{K}[\delta, \frac{d}{dt}]$. This is why we may also introduce matrices over $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ for some given π in $\mathfrak{K}[\delta]$. The corresponding module of matrices of size $p \times q$ will be denoted by $\mathcal{M}_{p,q}[\delta, \pi^{-1}, \frac{d}{dt}]$. More precisely, a matrix M belongs to $\mathcal{M}_{p,q}[\delta, \pi^{-1}, \frac{d}{dt}]$ if and only if there exists a finite $s \in \mathbb{N}$ such that $\pi^s \cdot M \in \mathcal{M}_{p,q}[\delta, \frac{d}{dt}]$.

Remark 1. It may be argued that the previous expression of u in function of x_1 is not feasible since $(1 - \delta)^{-1} = \sum_{j=0}^{+\infty} \delta^j$ implies

$$u(t) = \sum_{j=-2}^{+\infty} \left(-\frac{\dot{k}(t - j\tau)}{k^2(t - j\tau)} \dot{x}_1(t - j\tau) + \frac{1}{k(t - j\tau)} \ddot{x}_1(t - j\tau) \right)$$

which involves an infinite number of delayed terms. However, if we deal with motion planning, if x_1 is chosen constant outside the interval $[t_0, t_1]$, for some

³It is also often denoted by $GL_n(\mathfrak{K}[\delta, \frac{d}{dt}])$

$t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, then \dot{x}_1 and \ddot{x}_1 are identically equal to zero before t_0 and after t_1 , and the above series has at most $\lceil \frac{t_1 - t_0}{\tau} \rceil + 2$ non vanishing terms, where the notation $\lceil r \rceil$ stands for the least integer upper bound of an arbitrary non negative real r .

Unfortunately, $\mathfrak{K}[\delta, \frac{d}{dt}]$ and $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ are not Principal Ideal Domains (see e.g. [21, 32, 13, 5]), a property which is essential for our purpose (see the Smith-Jacobson decomposition in Appendix B). However, if we extend the ground field \mathfrak{K} to the fraction field $\mathfrak{K}(\delta)$, $\mathfrak{K}(\delta)[\frac{d}{dt}]$ is a principal ideal ring of polynomials of $\frac{d}{dt}$. We then construct the modules $\mathcal{M}_{p,q}(\delta)[\frac{d}{dt}]$ of matrices of size $p \times q$ and $\mathcal{U}_p(\delta)[\frac{d}{dt}]$ of unimodular matrices of size $p \times p$ respectively, over $\mathfrak{K}(\delta)[\frac{d}{dt}]$. Note that $\mathfrak{K}(\delta)[\frac{d}{dt}]$ strictly contains $\mathfrak{K}[\delta, \frac{d}{dt}]$ and $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ for every $\pi \in \mathfrak{K}[\delta]$. Therefore, to interpret the results of computations in $\mathfrak{K}(\delta)[\frac{d}{dt}]$, which turn out to be quite simple, and to decide if they belong to a suitable $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$, following [22], we have recourse to the notion of *localization* introduced in subsection 2.1.1. This aspect will be discussed in section 3.

Since $\mathfrak{K}(\delta)[\frac{d}{dt}]$ is a Principal Ideal Domain, the matrices of $\mathcal{M}_{p,q}(\delta)[\frac{d}{dt}]$ enjoy the essential property of admitting a so-called *Smith-Jacobson decomposition*⁴, or *diagonal decomposition*:

Theorem 1 (Smith-Jacobson decomposition [5, 13]). *Let $M \in \mathcal{M}_{p,q}(\delta)[\frac{d}{dt}]$ be an arbitrary polynomial matrix of size $p \times q$. There exist unimodular matrices $U \in \mathcal{U}_p(\delta)[\frac{d}{dt}]$ and $V \in \mathcal{U}_q(\delta)[\frac{d}{dt}]$ such that:*

$$UMV = \begin{cases} (\Delta_p | 0_{p,q-p}) & \text{if } p \leq q \\ \begin{pmatrix} \Delta_q \\ 0_{p-q,q} \end{pmatrix} & \text{if } p > q. \end{cases} \quad (5)$$

In both cases, $\Delta_\sigma \in \mathcal{M}_{\sigma,\sigma}(\delta)[\frac{d}{dt}]$, $\sigma = p$ or q , is a diagonal matrix whose diagonal elements $(d_1, \dots, d_s, 0, \dots, 0)$, with $s \leq \sigma$, are such that d_i is a nonzero $\frac{d}{dt}$ -polynomial for $i = 1, \dots, s$, with coefficients in $\mathfrak{K}(\delta)$, and is a divisor of d_j for all $1 \leq j \leq i$.

A constructive algorithm to compute this decomposition may be found in Section B of the Appendix.

Given an arbitrary matrix M , we call $\text{l-SJ}(M)$ (resp. $\text{r-SJ}(M)$), the left (resp. right) Smith-Jacobson subset of unimodular matrices $U \in \mathcal{U}_p(\delta)[\frac{d}{dt}]$ (resp. $V \in \mathcal{U}_q(\delta)[\frac{d}{dt}]$) such that there exists $V \in \mathcal{U}_q(\delta)[\frac{d}{dt}]$ (resp. $U \in \mathcal{U}_p(\delta)[\frac{d}{dt}]$) satisfying the decomposition (5).

⁴we adopt here the names of Smith and Jacobson for the diagonal decomposition to remind that it is credited to Smith [11, 14] in the commutative context and Jacobson [13, 3] for general principal ideal domains.

Example 2. Consider again system (3). The Smith-Jacobson decomposition of B is straightforward:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V = 1, \quad \Delta_1 = \delta, \quad UBV = \begin{pmatrix} \delta \\ 0 \end{pmatrix}$$

If we want to eliminate u in (3), using the previous Smith-Jacobson decomposition of B , we first remark that the second line of U , which will be denoted by $U_2 = (1 \ 0)$, corresponds to the left projection operator on the kernel of B , i.e. $U_2B = 0$. It suffices then to left multiply A by U_2 to obtain the implicit form

$$F(\delta, \frac{d}{dt})x \triangleq U_2Ax = \begin{pmatrix} \frac{d}{dt} & -k\delta(1-\delta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = U_2Bu = 0 \quad (6)$$

We may also compute a Smith-Jacobson decomposition of F : we first right multiply F by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to shift the 0-th order term in $\frac{d}{dt}$ to the left, yielding

$$F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -k\delta(1-\delta) & \frac{d}{dt} \end{pmatrix}$$

and then, right multiplying the result by

$$\begin{pmatrix} -\delta^{-1}(1-\delta)^{-1}\frac{1}{k} & \delta^{-1}(1-\delta)^{-1}\frac{1}{k}\frac{d}{dt} \\ 0 & 1 \end{pmatrix}$$

leads to $(1 \ 0)$. The Smith-Jacobson decomposition of F is therefore given by

$$U_F F V_F = (1 \ 0). \quad (7)$$

with

$$\begin{aligned} V_F &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\delta^{-1}(1-\delta)^{-1}\frac{1}{k} & \delta^{-1}(1-\delta)^{-1}\frac{1}{k}\frac{d}{dt} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\delta^{-1}(1-\delta)^{-1}\frac{1}{k} & \delta^{-1}(1-\delta)^{-1}\frac{1}{k}\frac{d}{dt} \end{pmatrix} \end{aligned} \quad (8)$$

and $U_F = 1$.

As previously discussed, the Smith-Jacobson decomposition has been computed over the ring $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ but, according to (8), its result may be expressed in the ring $\mathfrak{K}[\delta, \pi_F^{-1}, \frac{d}{dt}]$ with $\pi_F = (1-\delta)\delta$.

It is also easy to verify that, since the matrix F is a presentation matrix of the system module over $\mathfrak{K}[\delta, \pi_F^{-1}, \frac{d}{dt}]$, the latter module, according to (7), is isomorphic to any free finitely generated module admitting the matrix $(1 \ 0)$ as presentation matrix, which implies that the localized system module is free (see Section 2.1.1). Note also that the polynomial π_F admits non zero roots: every τ -periodic non zero meromorphic function f of the variable t satisfies $\pi_F f(t) = f(t-\tau) - f(t-2\tau) = 0$. Therefore, the system module is not $\mathfrak{K}[\delta, \frac{d}{dt}]$ -free.

We now introduce a remarkable class of matrices of $\mathcal{M}_{p,q}(\delta) \left[\frac{d}{dt} \right]$ called *hyper-regular*.

Definition 1 (Hyper-regularity [16]). *Given a matrix $M \in \mathcal{M}_{p,q}(\delta) \left[\frac{d}{dt} \right]$, we say that M is $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular, or simply hyper-regular if the context is non ambiguous, if and only if, in (5), $\Delta_p = I_p$ if $p \leq q$ (resp. $\Delta_q = I_q$ if $p > q$).*

Remark 2. *It is not difficult to prove that a finitely generated module Λ over the ring $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ whose presentation matrix F is hyper-regular cannot have torsion elements and is therefore free, since $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ is a Principal Ideal Ring. However, the system module Λ , over $\mathfrak{K}[\delta, \frac{d}{dt}]$, does not need to be free, as shown in the previous example. Nevertheless, it will be seen later that there exists a liberation polynomial π , deduced from the Smith-Jacobson decomposition of F , such that the $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ -system module, associated to the same presentation matrix F , is free.*

Example 3. *Going back to the decomposition of Example 2, a variant of this decomposition may be obtained as:*

$$U' = \begin{pmatrix} 0 & \delta^{-1} \\ 1 & 0 \end{pmatrix}, \quad V = 1, \quad \Delta'_1 = I_1 = 1 \quad (9)$$

which proves that B is $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular.

It is also immediate from (7) that F is $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular, or more precisely, $\mathfrak{K}[\delta, \pi_F^{-1}, \frac{d}{dt}]$ -hyper-regular with $\pi_F = (1 - \delta)\delta$. Note, on the contrary, that A is not $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular, though $F = U_2 A$ is. It is easily verified that

$$U_A A V_A = \Delta_A$$

with

$$\begin{aligned} U_A &= \begin{pmatrix} 1 & 0 \\ -\frac{k}{k} + \frac{d}{dt} & k\delta(1 - \delta) \end{pmatrix} \\ V_A &= \begin{pmatrix} 0 & 1 \\ -(1 - \delta)^{-1}\delta^{-1}\frac{1}{k} & (1 - \delta)^{-1}\delta^{-1}\frac{1}{k}\frac{d}{dt} \end{pmatrix} \end{aligned}$$

and

$$\Delta_A = \begin{pmatrix} 1 & 0 \\ 0 & (-\frac{k}{k} + \frac{d}{dt})\frac{d}{dt} \end{pmatrix}.$$

Thus, since the second degree $\frac{d}{dt}$ -polynomial $(-\frac{k}{k} + \frac{d}{dt})\frac{d}{dt}$ of the diagonal of Δ_A cannot be reduced, A is not hyper-regular.

2.1.3. Implicit system representation

One of the applications of the Smith-Jacobson decomposition concerns the possibility of expressing the system (1) in implicit form by eliminating the input u , which may be useful to work with a smaller number of variables.

For simplicity's sake, we rewrite system (1) $Ax = Bu$.

Proposition 1. *System (1) is equivalent to*

$$Fx = 0, \quad \Delta_B N^{-1}u = (I_m, 0_{m,n-m})MAx \quad (10)$$

with

$$F = (0_{n-m,m}, I_{n-m})MA, \quad (11)$$

$M \in \text{I-SJ}(B)$ and N such that

$$MBN = \begin{pmatrix} \Delta_B \\ 0_{n-m,m} \end{pmatrix}. \quad (12)$$

Moreover, if B is $\mathfrak{K}(\delta) \left[\frac{d}{dt}\right]$ -hyper-regular, the explicit form (1) admits the implicit representation

$$Fx = 0 \quad (13)$$

with F given by (11), and with $\Delta_B = I_m$ in (12). In this case, u is deduced from x by

$$u = N(I_m, 0_{m,n-m})MAx. \quad (14)$$

Proof. Consider a pair of matrices M and N obtained from the Smith-Jacobson decomposition of B , i.e. satisfying (12). Thus, left-multiplying both sides of system (1) by $(0_{n-m,m}, I_{n-m})M$, according to (11) we get $Fx = 0$. On the other hand, multiplying both sides of (1) by $(I_m, 0_{m,n-m})M$ we get

$$(I_m, 0_{m,n-m})MAx = \Delta_B N^{-1}u,$$

hence the representation (10).

Conversely, if x and u are given by (10), we have

$$MAx = \begin{pmatrix} (I_m, 0_{m,n-m})MA \\ (0_{n-m,m}, I_{n-m})MA \end{pmatrix} x = \begin{pmatrix} \Delta_B N^{-1}u \\ 0 \end{pmatrix} = MBu$$

the last equality being a consequence of (12). Thus, since M is unimodular, the pair (x, u) satisfies $Ax = Bu$, which proves the equivalence.

Finally, if B is $\mathfrak{K}(\delta) \left[\frac{d}{dt}\right]$ -hyper-regular, one can replace Δ_B by I_m and the second equation of (10) becomes (14). Thus, u is a $\mathfrak{K}(\delta) \left[\frac{d}{dt}\right]$ -combination of the components of x and can be eliminated. Therefore, the remaining part (13) is the desired implicit representation of (1). The proposition is proven. \square

In the sequel, if B is hyper-regular, we refer to (13) as the implicit representation of system (1).

Proposition 2. *For system (1) to be $\mathfrak{K}(\delta) \left[\frac{d}{dt}\right]$ -torsion free controllable (see [22, 31]), it is necessary that B and F , defined by (11), are $\mathfrak{K}(\delta) \left[\frac{d}{dt}\right]$ -hyper-regular.*

Moreover, in this case, there exists a polynomial $\bar{\pi}$ such that the localized system module at the powers of $\bar{\pi}$ is free.

Proof. Assume that the system is $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ torsion free controllable and that B is not $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular. Then, using the decomposition (12), Δ_B has at least one diagonal element which is a polynomial of degree larger than or equal to 1 with respect to $\frac{d}{dt}$ and with coefficients in $\mathfrak{K}(\delta)$. There indeed exists a non zero element v of the $\mathfrak{K}[\delta, \frac{d}{dt}]$ -free module generated by the components of u , such that $\Delta_B v = 0$ (v is a non trivial solution of a differential delay equation). It is immediately seen that the pair $x = 0, u = Nv$ is a non zero torsion element of the system module, which contradicts the freeness assumption.

If F is not $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular, its decomposition is given by $UF\tilde{Q} = (\Delta_F, 0)$, Δ_F having at least one diagonal element which is a polynomial of degree larger than or equal to 1 with respect to $\frac{d}{dt}$ and with coefficients in $\mathfrak{K}(\delta)$ which shows, using the representation (10), that every pair (ξ_1, ξ_2) such that $\Delta_F \xi_1 = 0$, $\xi_1 \neq 0$, and ξ_2 arbitrary, satisfies $(\Delta_F, 0) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$. Thus, the pair (x, u) with $x = \tilde{Q} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and u satisfying $\Delta_B N^{-1}u = (I_m, 0_{m, n-m})MA\tilde{Q} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ (see (10)) is a torsion element of the system module. Consequently, the latter module cannot be $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -free.

To prove the existence of $\bar{\pi}$, we remark that, according to the Smith-Jacobson decomposition algorithm (see Section B of the Appendix), if $M \in \text{I-SJ}(B)$ and $N \in \text{r-SJ}(B)$ are such that $MBN = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$, each row of M and N may contain the inverse of a polynomial of $\mathfrak{K}[\delta]$. Taking the LCM, say $\pi_{M,N}$, of these polynomials for all rows, we immediately get that $\pi_{M,N} \cdot M \in \mathcal{U}_n[\delta, \frac{d}{dt}]$, $\pi_{M,N} \cdot N \in \mathcal{U}_m[\delta, \frac{d}{dt}]$ with $\pi_{M,N} \in \mathfrak{K}[\delta]$. The same argument applies for F : consider $U \in \mathcal{U}_{n-m}(\delta) \left[\frac{d}{dt} \right]$, $\tilde{Q} \in \mathcal{U}_n(\delta) \left[\frac{d}{dt} \right]$ as above, namely such that $UF\tilde{Q} = (I_{n-m} \ 0_{n-m, m})$. There exists $\pi_{U, \tilde{Q}} \in \mathfrak{K}[\delta]$ such that $\pi_{U, \tilde{Q}} \cdot U \in \mathcal{U}_{n-m}[\delta, \frac{d}{dt}]$, $\pi_{U, \tilde{Q}} \cdot \tilde{Q} \in \mathcal{U}_n[\delta, \frac{d}{dt}]$. Taking $\bar{\pi}$ as the LCM of $\pi_{M,N}$ and $\pi_{U, \tilde{Q}}$, it is immediately seen from the decompositions of B and F , multiplied by suitable powers of $\bar{\pi}$, that the system module over the localized ring $\mathfrak{K}[\delta, \bar{\pi}^{-1}, \frac{d}{dt}]$ is free, and the proof is complete. \square

2.2. Differential π -Flatness

We first recall the classical definition of a flat system [16, 33], in the context of systems described by ordinary nonlinear differential equations: a system is said to be differentially flat if and only if there exists a set of independent variables, referred to as a flat output, such that every system variable (including the input variables) is a function of the flat output and a finite number of its successive time derivatives. More precisely, the system

$$\dot{x} = f(x, u)$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is differentially flat if and only if there exist a set of independent variables (flat output)

$$y = h(x, u, \dot{u}, \ddot{u}, \dots, u^{(r)}), \quad y \in \mathbb{R}^m \quad (15)$$

such that

$$x = \alpha(y, \dot{y}, \ddot{y}, \dots, y^{(s)}) \quad (16)$$

$$u = \beta(y, \dot{y}, \ddot{y}, \dots, y^{(s+1)}) \quad (17)$$

and such that the system equations

$$\frac{d\alpha}{dt}(y, \dot{y}, \ddot{y}, \dots, y^{(s+1)}) = f\left(\alpha(y, \dot{y}, \ddot{y}, \dots, y^{(s)}), \beta(y, \dot{y}, \ddot{y}, \dots, y^{(s+1)})\right) \quad (18)$$

are identically satisfied for all smooth enough function $t \mapsto y(t)$, r and s being suitable finite m -tuples of integers.

Let us now discuss further extensions of this definition to linear delay systems and consider first an elementary example to motivate the next definition.

Example 4. *Let us consider the elementary system*

$$\dot{x}(t) = u(t - \tau)$$

with $A = \frac{d}{dt}$ and $B = \delta$ in the notations of (1). Clearly, if we set $y = x$, y looks like a flat output though $u = \delta^{-1}y$ contains a one-step prediction and belongs to $\mathfrak{K}(\delta)[\frac{d}{dt}]$ but not to $\mathfrak{K}[\delta, \frac{d}{dt}]$. However, for motion planning, such a dependence remains acceptable (see Remark 1), even if for feedback design it poses more delicate problems. This notion is called differential δ -flatness (see e.g. [22, 31]).

According to [22], this notion is generalized as follows:

Definition 2 (Differential π -flatness [22]). *The linear delay system (1) is said to be differentially π -flat (or π -free) if and only if there exists a polynomial $\pi \in \mathfrak{K}[\delta]$, and a collection y of m (δ, π^{-1}) -differentially independent variables⁵, called π -flat output, of the form*

$$y = P_0(\delta, \pi^{-1})x + P_1(\delta, \pi^{-1}, \frac{d}{dt})u, \quad (19)$$

with $P_0(\delta, \pi^{-1}) \in (\mathfrak{K}[\delta, \pi^{-1}])^{m \times n}$ the set of matrices of size $m \times n$, with coefficients in $\mathfrak{K}[\delta, \pi^{-1}]$, $P_1(\delta, \pi^{-1}, \frac{d}{dt}) \in \mathcal{M}_{m,m}[\delta, \pi^{-1}, \frac{d}{dt}]$, and such that

$$x = Q(\delta, \pi^{-1}, \frac{d}{dt})y, \quad (20)$$

$$u = R(\delta, \pi^{-1}, \frac{d}{dt})y, \quad (21)$$

with $Q(\delta, \pi^{-1}, \frac{d}{dt}) \in \mathcal{M}_{n,m}[\delta, \pi^{-1}, \frac{d}{dt}]$, and $R(\delta, \pi^{-1}, \frac{d}{dt}) \in \mathcal{M}_{m,m}[\delta, \pi^{-1}, \frac{d}{dt}]$.

⁵more precisely, there does not exist a non zero matrix $S \in \mathcal{M}_{m,m}[\delta, \pi^{-1}, \frac{d}{dt}]$ such that $Sy = 0$, or equivalently, $Sy = 0$ implies $S \equiv 0$

In other words, definition 2 states that the components of a π -flat output y can be obtained as a $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ -linear combination of the system variables, and that the system variables (x, u) are also $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ -linear combinations of the components of y . Thus x and u can be calculated from y using differentiations, delays, and predictions (coming from the inverse of π). Note that the facts that every element of the system module is a $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ -linear combination of the components of y , and that the components of y are $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ -independent, indeed imply that the components of y form a basis of the system module $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}] \otimes_{\mathfrak{K}[\delta, \frac{d}{dt}]} \Lambda$, which is therefore free, hence the equivalence with the definition of subsection 2.1.1.

If system (1) is considered in implicit form (13) after elimination of the input u , since this elimination expresses u as a $\mathfrak{K}(\delta) [\frac{d}{dt}]$ -combination of the components of x , the expression (19), combined with (14), reads $y = Px$ with $P \in \mathcal{M}_{m,n}(\delta) [\frac{d}{dt}]$. The previous definition is thus adapted as follows:

Definition 3. *The implicit linear delay system (13) is said to be differentially π -flat if and only if there exists a polynomial $\pi \in \mathfrak{K}[\delta]$, and a collection y of m (δ, π^{-1}) -differentially independent variables, called π -flat output, of the form*

$$y = P(\delta, \pi^{-1}, \frac{d}{dt})x, \quad (22)$$

with $P(\delta, \pi^{-1}, \frac{d}{dt}) \in \mathcal{M}_{m,n}[\delta, \pi^{-1}, \frac{d}{dt}]$, and such that

$$x = Q(\delta, \pi^{-1}, \frac{d}{dt})y, \quad (23)$$

with $Q(\delta, \pi^{-1}, \frac{d}{dt}) \in \mathcal{M}_{n,m}[\delta, \pi^{-1}, \frac{d}{dt}]$.

The matrices P , Q and R of (19)–(21) in the explicit case, and P and Q of (22)–(23) in the implicit case, are called *defining operators* of the π -flat output y .

Remark 3. *In [22] and later (see e.g. [31]), the above notion is often called π -freeness and introduced via the notion of system module. The wording π -flatness appears, to the authors knowledge, for the first time in [24]. It has also been related to system parameterization in [28, 4]. We have preferred here the name π -flatness, in reference to differential flatness, and to directly present it via the notion of flat output, rather than basis of the system module. Note that in formula (19), P_0 is a 0th degree polynomial of $\frac{d}{dt}$ to mimic the general definition in (15) that does not include time derivatives of x , with (19)–(21) restricted to linear expressions.*

Example 5. *Let us go back again to example 1 and let us prove that $y = x_1$ is a π -flat output with $\pi = (1 - \delta)\delta^2$. From (4), we have*

$$x_2 = \delta^{-1}(1 - \delta)^{-1} \frac{1}{k} \dot{x}_1 = \delta^{-1}(1 - \delta)^{-1} \frac{1}{k} \dot{y}$$

$$u = \delta^{-1} \frac{d}{dt} x_2 = \delta^{-2} (1 - \delta)^{-1} \left(-\frac{\dot{k}}{k^2} \dot{y} + \frac{1}{k} \ddot{y} \right) \quad (24)$$

In other words, following the notations of (19)–(21), $P_0 = (1, 0)$, $P_1 = 0$ and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \pi^{-1} \delta \frac{1}{k} \frac{d}{dt} \end{pmatrix} y \triangleq Q(\delta, \pi^{-1}, \frac{d}{dt}) y \quad (25)$$

$$u = \pi^{-1} \left(-\frac{\dot{k}}{k^2} \frac{d}{dt} + \frac{1}{k} \frac{d^2}{dt^2} \right) y \triangleq R(\delta, \pi^{-1}, \frac{d}{dt}) y \quad (26)$$

which proves that y is a π -flat output.

3. Main Result

In this section, we propose a simple and effective algorithm for the computation of π -flat outputs of linear time-delay systems based on the following necessary and sufficient condition for the existence of defining operators of a π -flat output. Moreover, explicit expressions of π and of these operators are obtained.

Theorem 2. *A necessary and sufficient condition for system (1) to be π -flat is that the matrices B and F are $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular.*

We construct the operators P , Q and R and the polynomial π as follows.

0. According to Propositions 1 and 2, construct the decomposition of B (12), define F by (11) and compute $\bar{\pi}$;
1. $Q = \tilde{Q} \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$, with $\tilde{Q} \in \text{r-SJ}(F)$. Note that $\bar{\pi} \cdot Q \in \mathcal{M}_{n,m}[\delta, \frac{d}{dt}]$ by construction;
2. $R = N(I_m, 0_{m,n-m})MAQ$, with $N \in \text{r-SJ}(B)$. There exists $\pi_R \in \mathfrak{K}[\delta]$ such that $\pi_R \cdot R \in \mathcal{M}_{m,m}[\delta, \frac{d}{dt}]$;
3. $P = W(I_m, 0_{m,n-m})\tilde{P}$, with $\tilde{P} \in \text{l-SJ}(Q)$ and $W \in \text{r-SJ}(Q)$. There exists $\pi_P \in \mathfrak{K}[\delta]$ such that $\pi_P \cdot P \in \mathcal{M}_{m,n}[\delta, \frac{d}{dt}]$.

Let π be given by $\pi = \text{LCM}(\bar{\pi}, \pi_P, \pi_R)$, the least common multiple of $\bar{\pi}$, π_P and π_R .

Thus, P , Q and R are defining operators with $y = Px$, $x = Qy$, $u = Ry$, and y is a π -flat output.

Proof. If B or F are not $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regular, according to Proposition 2, the $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ system module cannot be torsion free. Therefore, system (1) cannot be π -flat, for any $\pi \in \mathfrak{K}[\delta]$. Taking the contrapositive of this statement, the hyper-regularity of B and F is proven to be necessary.

We now prove that the $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$ -hyper-regularity of B and F is sufficient to construct the defining matrices P , Q and R as well as a liberation polynomial π .

We first use the implicit form (13) of Proposition 1, and more precisely (11)–(14) to obtain a decomposition of B and F . Since F is hyper-regular by assumption, there exist $U \in \mathcal{U}_{n-m}(\delta) \left[\frac{d}{dt} \right]$, $\tilde{Q} \in \mathcal{U}_n(\delta) \left[\frac{d}{dt} \right]$ such that $UF\tilde{Q} = (I_{n-m} \ 0_{n-m,m})$. Consequently $Q = \tilde{Q} \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$, with $\tilde{Q} \in \text{r-SJ}(F)$, is such that $FQ = 0_{n-m,m}$. Thus setting $x = Qy$ we have that $FQy = 0$ for all y .

The existence of $\bar{\pi} \in \mathfrak{K}[\delta]$ is proved following the same argument as in the proof of Proposition 2: for an arbitrary $\alpha \times \beta$ hyper-regular matrix M , according to the Smith-Jacobson decomposition algorithm (see Section B of the Appendix), if $U_M \in \text{l-SJ}(M)$ and $V_M \in \text{r-SJ}(M)$ are such that $U_M M V_M = (I_\alpha, 0)$ if $\alpha \leq \beta$ (resp. $U_M M V_M = \begin{pmatrix} I_\beta \\ 0 \end{pmatrix}$ if $\alpha \geq \beta$), each row of U_M and V_M may contain the inverse of a polynomial of $\mathfrak{K}[\delta]$. Taking the LCM, say π_M , of these polynomials for all rows, we immediately get that $\pi_M \cdot U_M \in \mathcal{M}_{\alpha,\beta}[\delta, \frac{d}{dt}]$ with $\pi_M \in \mathfrak{K}[\delta]$. Applying this result to the decompositions of B and F , we have proven the existence of $\bar{\pi}$ such that $\bar{\pi} \cdot \tilde{Q} \in \mathcal{U}_n[\delta, \frac{d}{dt}]$ and $\bar{\pi} \cdot Q \in \mathcal{M}_{n,m}[\delta, \frac{d}{dt}]$, which proves item 1.

Going back to (12) and (14), setting $R = N(I_m \ 0_{m,n-m})MAQ$, we obtain $u = Ry$ with $N \in \text{r-SJ}(B)$. Finally, the proof of the existence of $\pi_R \in \mathfrak{K}[\delta]$ such that $\pi_R \cdot R \in \mathcal{M}_{m,m}[\delta, \frac{d}{dt}]$ follows the same lines as in item 1, which proves 2.

Since Q is hyper-regular by construction, its Smith-Jacobson decomposition yields the existence of $\tilde{P} \in \mathcal{U}_n(\delta) \left[\frac{d}{dt} \right]$ and $W \in \mathcal{U}_m(\delta) \left[\frac{d}{dt} \right]$ such that

$$\tilde{P}QW = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix}$$

thus $(I_m \ 0_{m,n-m})\tilde{P}QW = I_m$ and, setting

$$P = W(I_m, 0_{m,n-m})\tilde{P}$$

it results that $Px = W(I_m, 0_{m,n-m})\tilde{P}x = W(I_m, 0_{m,n-m})\tilde{P}Qy = WW^{-1}y = y$, and the proof of the third item is complete, noting again that the existence of $\pi_P \in \mathfrak{K}[\delta]$ such that $\pi_P \cdot P \in \mathcal{M}_{m,n}[\delta, \frac{d}{dt}]$ follows the same lines as in items 1 and 2.

Finally, taking $\pi = \text{LCM}(\bar{\pi}, \pi_P, \pi_R)$, the least common multiple of $\bar{\pi}$, π_P and π_R , it is straightforward to show that $\pi \cdot P \in \mathcal{M}_{m,n}[\delta, \frac{d}{dt}]$, $\pi \cdot Q \in \mathcal{M}_{n,m}[\delta, \frac{d}{dt}]$ and $\pi \cdot R \in \mathcal{M}_{m,m}[\delta, \frac{d}{dt}]$. Therefore, P , Q and R are defining matrices of a π -flat output for system (1) and thus that system (1) is π -flat. \square

Theorem 2 is easily translated into the Algorithm 1 presented below.

Remark 4. The π -flatness criterion is given in terms of properties of the matrices B and F that depend only on the larger ring $\mathfrak{K}(\delta) \left[\frac{d}{dt} \right]$. If, in addition,

Algorithm 1: Procedure to compute π -flat outputs.

Input: Two matrices $A \in \mathcal{M}_{n,n}[\delta, \frac{d}{dt}]$ and $B \in \mathcal{M}_{n,m}[\delta, \frac{d}{dt}]$.

Output: a polynomial $\pi \in \mathfrak{K}[\delta]$ and defining operators

$P \in \mathcal{M}_{m,n}[\delta, \pi^{-1}, \frac{d}{dt}]$, $Q \in \mathcal{M}_{n,m}[\delta, \pi^{-1}, \frac{d}{dt}]$ and

$R \in \mathcal{M}_{m,m}[\delta, \pi^{-1}, \frac{d}{dt}]$ such that $y = Px$, $x = Qy$ and $u = Ry$.

Initialization: Test of hyper-regularity of B by its Smith-Jacobson decomposition, which also provides $M \in \text{l-SJ}(B)$ and $N \in \text{r-SJ}(B)$, i.e. such that $MBN = (I_m, 0_{n-m,m})^T$. If B is not hyper-regular, the system is not π -flat whatever $\pi \in \mathfrak{K}[\delta]$.

Algorithm:

- 1 Set $F = (0_{n-m,m}, I_{n-m})MA$ and test if F is hyper-regular by computing its Smith-Jacobson decomposition: $VF\tilde{Q} = (I_{n-m}, 0_{n-m,m})$. If F is not hyper-regular, the system is not π -flat whatever $\pi \in \mathfrak{K}[\delta]$. Otherwise, compute $\tilde{\pi} \in \mathfrak{K}[\delta]$ such that $\tilde{\pi} \cdot M \in \mathcal{U}_n[\delta, \frac{d}{dt}]$, $\tilde{\pi} \cdot N \in \mathcal{U}_m[\delta, \frac{d}{dt}]$, $\tilde{\pi} \cdot \tilde{Q} \in \mathcal{U}_n[\delta, \frac{d}{dt}]$.
 - 2 Compute $Q = \tilde{Q}(0_{n-m,m}, I_m)^T$, $R = N(I_m, 0_{m,n-m})MAQ$ and $\pi_R \in \mathfrak{K}[\delta]$ such that $\pi_R \cdot R \in \mathcal{M}_{m,m}[\delta, \frac{d}{dt}]$.
 - 3 Compute a Smith-Jacobson decomposition of Q : $\tilde{P}QW = (I_m, 0_{n-m,m})^T$, $P = W(I_m, 0_{m,n-m})\tilde{P}$, and a polynomial $\pi_P \in \mathfrak{K}[\delta]$ such that $\pi_P \cdot P \in \mathcal{M}_{m,n}[\delta, \frac{d}{dt}]$.
 - 4 Compute the polynomial $\pi = \text{LCM}(\tilde{\pi}, \pi_P, \pi_R) \in \mathfrak{K}[\delta]$. The system is π -flat.
-

the submodule of $\mathfrak{K}[\delta]$ generated by the powers of π is torsion free (e.g. $\pi = \delta$, for which the equation $\delta f = 0$ admits the unique solution $f = 0$), then the system module Λ over $\mathfrak{K}[\delta, \pi^{-1}, \frac{d}{dt}]$ is torsion free, and the original module Λ (over $\mathfrak{K}[\delta, \frac{d}{dt}]$) is free if, and only if, $\pi = 1$. Note that computing a free basis of the system module directly over $\mathfrak{K}[\delta, \frac{d}{dt}]$, would require more elaborated tools such as those developed in [28, 27, 29].

Remark 5. Since the computations are made in the larger modules $\mathcal{M}_{p,q}(\delta)[\frac{d}{dt}]$ for suitable p and q , Theorem 2 remains valid for systems depending on an arbitrary but finite number of delays, say $\delta_1, \dots, \delta_s$, by replacing the field $\mathfrak{K}(\delta)$, on which $\mathcal{M}_{p,q}(\delta)[\frac{d}{dt}]$ is modeled, by the fraction field $\mathfrak{K}(\delta_1, \dots, \delta_s)$ generated by the multivariate polynomials of $\delta_1, \dots, \delta_s$. An example with two independent delays is presented in Example 4.3.

Remark 6. Our results also apply in the particular case of linear time-varying systems without delays. If \mathfrak{K} is the field of meromorphic functions of time, $\mathfrak{K}[\frac{d}{dt}]$ is a Principal Ideal Domain and all the computations involved in Theorem 2 and the associated algorithm, remain in this ring, contrarily to the case with delay. Therefore, the last step consisting in finding the so-called liberation polynomial π is needless. Related results may be found in [18, 35, 28].

4. Examples

4.1. Back to the Introductory Example

Going back to Example 1, let us apply the previous algorithm to the time-delay system defined by (3), for which a π -flat output is already known from Example 5. The first step, consisting in the computation of a Smith-Jacobson decomposition of the matrix B has already been done, the left and right unimodular matrices $M \in \text{l-SJ}(B)$ and $N \in \text{r-SJ}(B)$ being given by (9) with $M = U'$ and $N = V = 1$. Then the matrix $F = (0 \ 1)MA$ of an implicit representation of (3) is given by (6). Its Smith-Jacobson decomposition $VF\tilde{Q} = (1 \ 0)$ is given by (7)-(8), with $V = U_F = 1$ and $\tilde{Q} = V_F$, and has been seen to be hyper-regular in Example 3. We easily check that $\bar{\pi} = (1 - \delta)\delta$.

Going on with step 2, we set

$$Q = \tilde{Q} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \delta^{-1}(1 - \delta)^{-1} \frac{1}{k(t)} \frac{d}{dt} \end{pmatrix}$$

and verify that $\bar{\pi} \cdot Q \in \mathcal{M}_{2,1}[\delta, \frac{d}{dt}]$. Moreover

$$R = N(1 \ 0)MAQ = \delta^{-2}(1 - \delta)^{-1} \left(-\frac{\dot{k}(t)}{k^2(t)} \frac{d}{dt} + \frac{1}{k(t)} \frac{d^2}{dt^2} \right)$$

and we have $\pi_R = \bar{\pi}\delta$.

Note that, setting $x = Qy$ and $u = Ry$, we recover formulae (25) and (24) (or equivalently (26)).

According to step 3 of the algorithm, we compute a Smith-Jacobson decomposition of Q : $\tilde{P}QW = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which provides $W = 1$ and

$$\tilde{P} = \begin{pmatrix} 1 \\ -\delta^{-1}(1 - \delta)^{-1} \frac{1}{k(t)} \frac{d}{dt} \end{pmatrix}$$

hence $P = (1 \ 0)$ and $y = Px = x_1$. Here, $\pi_P = 1$. Finally, the least common multiple of $(1 - \delta)\delta$, $(1 - \delta)\delta^2$ and 1 is $\pi = (1 - \delta)\delta^2$. We have thus verified that Algorithm 1 comes up with the same conclusion as Example 5.

4.2. A Multi-input Example

Let us consider the following academic example of multi-input delay system:

$$\begin{cases} \dot{x}_1(t) + x_1^{(2)}(t) - 2x_1^{(2)}(t - \tau) + x_1^{(3)}(t) + x_1^{(4)}(t - \tau) - x_2^{(3)}(t) + x_2^{(5)}(t) - x_3^{(2)}(t) \\ - \dot{x}_4(t) + x_4^{(3)}(t) = u_1(t) + \dot{u}_1(t) + u_2(t), \\ \dot{x}_1(t) + \dot{x}_1(t - \tau) - \dot{x}_1(t - 2\tau) + x_1^{(2)}(t) + x_1^{(2)}(t - \tau) + x_1^{(2)}(t - 2\tau) - x_1^{(3)}(t - \tau) \\ + 2\dot{x}_2(t) + \dot{x}_2(t - \tau) - x_2^{(2)}(t) - x_2^{(4)}(t) + \dot{x}_3(t) + x_3^{(2)}(t - \tau) - x_4(t) - x_4(t - \tau) \\ - x_4^{(2)}(t) = u_1^{(2)}(t - \tau) + \dot{u}_2(t - \tau), \\ -x_1(t - 2\tau) + \dot{x}_1(t - 3\tau) + x_1^{(2)}(t - 2\tau) - x_2(t - \tau) + \dot{x}_2(t - 2\tau) + x_2^{(3)}(t - \tau) \\ - x_3(t - \tau) + \dot{x}_3(t - 2\tau) + \dot{x}_4(t - \tau) = \dot{u}_1(t - 2\tau) + u_2(t - 2\tau), \\ \dot{x}_1(t - \tau) + \dot{x}_2(t) + \dot{x}_3(t) = \dot{u}_1(t) + u_2(t). \end{cases} \quad (27)$$

Denoting by x the state vector, $x = (x_1, x_2, x_3, x_4)^T$, by u the input vector, $u = (u_1, u_2)^T$, and by δ the delay operator of length τ , system (27) can be rewritten in matrix form $Ax = Bu$, with $A \in \mathcal{M}_{4,4}[\delta, \frac{d}{dt}]$ and $B \in \mathcal{M}_{4,2}[\delta, \frac{d}{dt}]$ defined by

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \end{pmatrix} \quad (28)$$

with

$$\begin{aligned} A_1 &= \begin{pmatrix} \frac{d}{dt} + \frac{d^2}{dt^2}(1 - 2\delta) + \frac{d^3}{dt^3} + \frac{d^4}{dt^4}\delta \\ \frac{d}{dt}(1 + \delta - \delta^2) + \frac{d^2}{dt^2}(1 + \delta + \delta^2) - \frac{d^3}{dt^3}\delta \\ -\delta^2 + \frac{d}{dt}\delta^3 + \frac{d^2}{dt^2}\delta^2 \\ \frac{d}{dt}\delta \end{pmatrix} \\ A_2 &= \begin{pmatrix} -\frac{d^3}{dt^3} + \frac{d^5}{dt^5} \\ \frac{d}{dt}(2 + \delta) - \frac{d^2}{dt^2} - \frac{d^4}{dt^4} \\ -\delta + \frac{d}{dt}\delta^2 + \frac{d^3}{dt^3}\delta \\ \frac{d}{dt} \end{pmatrix} \\ A_3 &= \begin{pmatrix} -\frac{d^2}{dt^2} \\ \frac{d}{dt} + \frac{d^2}{dt^2}\delta \\ -\delta + \frac{d}{dt}\delta^2 \\ \frac{d}{dt} \end{pmatrix}, \quad A_4 = \begin{pmatrix} -\frac{d}{dt} + \frac{d^3}{dt^3} \\ -(1 + \delta) - \frac{d^2}{dt^2} \\ \frac{d}{dt}\delta \\ 0 \end{pmatrix} \end{aligned} \quad (29)$$

and

$$B = \begin{pmatrix} 1 + \frac{d}{dt} & 1 \\ \frac{d^2}{dt^2}\delta & \frac{d}{dt}\delta \\ \frac{d}{dt}\delta^2 & \delta^2 \\ \frac{d}{dt} & 1 \end{pmatrix}.$$

We apply Algorithm 1 to compute a π -flat output if it exists. We start with the Smith-Jacobson decomposition of B . By left multiplying B by the following product of unimodular matrices $M = M_4 M_3 M_2 M_1 \in \mathcal{U}_4(\delta) [\frac{d}{dt}]$, given by

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{d^2}{dt^2}\delta & 1 & 0 & 0 \\ -\frac{d}{dt}\delta^2 & 0 & 1 & 0 \\ -\frac{d}{dt} & 0 & 0 & 1 \end{pmatrix}, \\ M_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{dt}\delta & 1 & 0 \\ 0 & -\delta^2 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and by setting $N = I_2$, we obtain the Smith-Jacobson decomposition

$$MBN = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -\frac{d}{dt} & 0 & 0 & \frac{d}{dt} + 1 \\ 0 & 1 & 0 & -\frac{d}{dt}\delta \\ 0 & 0 & 1 & -\delta^2 \end{pmatrix} \begin{pmatrix} \frac{d}{dt} + 1 & 1 \\ \frac{d^2}{dt^2}\delta & \frac{d}{dt}\delta \\ \frac{d}{dt}\delta^2 & \delta^2 \\ \frac{d}{dt} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

thus showing that B is hyper-regular. We then compute an implicit representation of (27) by $F = (0_{2,2} \quad I_2)MA$, i.e.

$$F = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix} \quad (30)$$

with

$$\begin{aligned} F_1 &= \begin{pmatrix} \frac{d}{dt}(1 + \delta - \delta^2) + \frac{d^2}{dt^2}(1 + \delta) - \frac{d^3}{dt^3}\delta \\ -\delta^2 + \frac{d^2}{dt^2}\delta^2 \end{pmatrix} \\ F_2 &= \begin{pmatrix} \frac{d}{dt}(2 + \delta) - \frac{d^2}{dt^2}(1 + \delta) - \frac{d^4}{dt^4} \\ -\delta + \frac{d^3}{dt^3}\delta \end{pmatrix} \\ F_3 &= \begin{pmatrix} \frac{d}{dt} \\ -\delta \end{pmatrix}, \quad F_4 = \begin{pmatrix} -1 - \delta - \frac{d^2}{dt^2} \\ \frac{d}{dt}\delta \end{pmatrix} \end{aligned} \quad (31)$$

to which corresponds the difference-differential system

$$\begin{aligned} &\left(\dot{x}_1(t) + \dot{x}_1(t - \tau) - \dot{x}_1(t - 2\tau) + \ddot{x}_1(t) + \ddot{x}_1(t - \tau) - x_1^{(3)}(t - \tau) \right) \\ &\quad + \left(2\dot{x}_2(t) + \dot{x}_2(t - \tau) - \ddot{x}_2(t) - \ddot{x}_2(t - \tau) - x_2^{(4)}(t) \right) \\ &\quad + \dot{x}_3(t) - (x_4(t) + x_4(t - \tau) + \ddot{x}_4(t)) = 0, \\ &(-x_1(t - 2\tau) + \ddot{x}_1(t - 2\tau)) - (x_2(t - \tau) - x_2^{(3)}(t - \tau)) \\ &\quad - x_3(t - \tau) + \dot{x}_4(t - \tau) = 0. \end{aligned}$$

According to step 1, we compute a right Smith-Jacobson decomposition of F :

$$VF\tilde{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (32)$$

where $V = 1$ and

$$\tilde{Q} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -(1 + \delta)^{-1} \frac{d}{dt} & -\delta^{-1}(1 + \delta)^{-1} \left(1 + \delta + \frac{d^2}{dt^2} \right) & \frac{d^2}{dt^2} - 1 & \frac{d^3}{dt^3} + \frac{d^2}{dt^2} - \delta \\ -(1 + \delta)^{-1} & -\delta^{-1}(1 + \delta)^{-1} \frac{d}{dt} & \left(1 - \frac{d}{dt} \right) \frac{d}{dt} & \left(\frac{d}{dt} + 1 - \delta \right) \frac{d}{dt} \end{pmatrix},$$

showing thus that F is hyper-regular and that $\bar{\pi} = \delta(1 + \delta)$.

For the interested reader, \tilde{Q} is obtained as the product $\tilde{Q} = \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3$ of matrices of elementary actions:

$$\begin{aligned} \tilde{Q}_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{d}{dt} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \tilde{Q}_2 &= \begin{pmatrix} \tilde{q}_{1,1,2} & \tilde{q}_{1,2,2} & \tilde{q}_{1,3,2} & \tilde{q}_{1,4,2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned}\tilde{q}_{1,1,2} &= -(1+\delta)^{-1} \\ \tilde{q}_{1,2,2} &= (1+\delta)^{-1} \frac{d}{dt} \\ \tilde{q}_{1,3,2} &= (1+\delta)^{-1} \left((2+\delta) \frac{d}{dt} - \frac{d^4}{dt^4} \right) - \frac{d^2}{dt^2} \\ \tilde{q}_{1,4,2} &= \frac{d}{dt} - \delta^2 (1+\delta)^{-1} \frac{d}{dt} + \frac{d^2}{dt^2} - (1+\delta)^{-1} \frac{d^3}{dt^3} \delta\end{aligned}\tag{33}$$

and

$$\tilde{Q}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\delta^{-1} & -1 + \frac{d^3}{dt^3} & \frac{d^2}{dt^2} \delta - \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

According to step 2, we get

$$Q = \tilde{Q} \begin{pmatrix} 0_{2,2} \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{d^2}{dt^2} - 1 & \frac{d^3}{dt^3} + \frac{d^2}{dt^2} - \delta \\ (1 - \frac{d}{dt}) \frac{d}{dt} & (\frac{d}{dt} + 1 - \delta) \frac{d}{dt} \end{pmatrix}$$

$$R = N(I_2 \quad 0_{2,2}) M A Q = \begin{pmatrix} -\frac{d^3}{dt^3} & \frac{d}{dt} - \frac{d^3}{dt^3} - \frac{d^4}{dt^4} \\ (\frac{d}{dt} + 1) \frac{d^3}{dt^3} & -\frac{d^2}{dt^2} + \frac{d^3}{dt^3} + 2 \frac{d^4}{dt^4} + \frac{d^5}{dt^5} \end{pmatrix},$$

with $\pi_R = 1$. From $x = Qy$ and $u = Ry$, we deduce the expressions

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ y_1(t) \\ -y_1(t) + y_1^{(2)}(t) - y_2(t - \tau) + y_2^{(2)}(t) + y_2^{(3)}(t) \\ \dot{y}_1(t) - y_1^{(2)}(t) + \dot{y}_2(t) - \dot{y}_2(t - \tau) + y_2^{(2)}(t) \end{pmatrix}.\tag{34}$$

and

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -y_1^{(3)}(t) + \dot{y}_2(t) - y_2^{(3)}(t) - y_2^{(4)}(t) \\ y_1^{(3)}(t) + y_1^{(4)}(t) - y_2^{(2)}(t) + y_2^{(3)}(t) + 2y_2^{(4)}(t) + y_2^{(5)}(t) \end{pmatrix}.\tag{35}$$

Next, according to step 3, we compute $\tilde{P} \in \mathcal{U}_4(\delta) [\frac{d}{dt}]$ and $W \in \mathcal{U}_2(\delta) [\frac{d}{dt}]$ such that $\tilde{P} Q W = (I_2 \quad 0_{2,2})^T$.

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{d^3}{dt^3} - \frac{d^2}{dt^2} + \delta & -\frac{d^2}{dt^2} + 1 & 1 & 0 \\ \frac{d\delta}{dt} - \frac{d^2}{dt^2} - \frac{d}{dt} & -\frac{d}{dt} + \frac{d^2}{dt^2} & 0 & 1 \end{pmatrix}, \quad W = I_2.$$

Again, \tilde{P} is obtained as the product $\tilde{P} = \tilde{P}_2 \tilde{P}_1$ of elementary actions:

$$\tilde{P}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{d^2}{dt^2} + 1 & -\frac{d^3}{dt^3} - \frac{d^2}{dt^2} + \delta & 1 & 0 \\ - (1 - \frac{d}{dt}) \frac{d}{dt} & - (\frac{d}{dt} + 1 - \delta) \frac{d}{dt} & 0 & 1 \end{pmatrix}, \quad \tilde{P}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$P = W \begin{pmatrix} I_2 & 0_{2,2} \end{pmatrix} \tilde{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

which, with $y = Px$, yields

$$y_1 = x_2, \quad y_2 = x_1 \quad (36)$$

and $\pi_P = 1$.

Then it is immediately seen that $\pi = \delta(1 + \delta)$ and that the system is π -flat.

Remark 7. *It is worth noting that the polynomial $\pi = \delta(1 + \delta)$ only appears at the intermediate level of the computation of \tilde{Q} , and not anymore in the defining matrices P , Q and R . However, this means that the system module contains elements z such that $z(t) = -z(t - \tau)$ for all t , that satisfy $\pi z = 0$, thus preventing this module from being free.*

4.3. Vibrating String With an Interior Mass

As noted in Remark 5, the computation of π -flat outputs based on Theorem 2 can be extended to linear systems with multiple delays. As an example, we consider the system of vibrating string with two controls proposed in [23], which can be modeled as a set of one-dimensional wave equations together with a second order linear ordinary differential equation describing the motion of the mass. Using Mikusiński operational calculus (see for instance [10]), this infinite-dimensional system can be transformed into the time-delay system

$$\begin{cases} \psi_1(t) + \phi_1(t) - \psi_2(t) - \phi_2(t) = 0, \\ \dot{\psi}_1(t) + \dot{\phi}_1(t) + \eta_1(\phi_1(t) - \psi_1(t)) - \eta_2(\phi_2(t) - \psi_2(t)) = 0, \\ \phi_1(t - 2\tau_1) + \psi_1(t) = u_1(t - \tau_1), \\ \phi_2(t) + \psi_2(t - 2\tau_2) = u_2(t - \tau_2). \end{cases} \quad (37)$$

where η_1 and η_2 are constant parameters. Denoting the state $x = (\psi_1, \phi_1, \psi_2, \phi_2)^T$, the control input $u = (u_1, u_2)$, and δ_1, δ_2 the delay operators of respective lengths τ_1 and τ_2 , the system (37) may be rewritten in the form $Ax = Bu$, with $A \in \mathcal{M}_{4,4}(\delta_1, \delta_2) \left[\frac{d}{dt} \right]$ and $B \in \mathcal{M}_{4,2}(\delta_1, \delta_2) \left[\frac{d}{dt} \right]$ given by

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ \frac{d}{dt} + \eta_1 & \frac{d}{dt} - \eta_1 & \eta_2 & -\eta_2 \\ 1 & \delta_1^2 & 0 & 0 \\ 0 & 0 & \delta_2^2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}. \quad (38)$$

The computation of a Smith-Jacobson decomposition of B is here straightforward: it suffices to exchange the two last lines of B with the two first lines, and we get $MBN = \begin{pmatrix} I_2 \\ 0_{2,2} \end{pmatrix}$ with

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta_1^{-1} & 0 & 0 & 0 \\ 0 & \delta_2^{-1} & 0 & 0 \end{pmatrix}, \quad N = I_2. \quad (39)$$

We have then $\pi_{M,N} = \delta_1 \delta_2$. Thus B is hyper-regular and

$$F = (0_{2,2} \quad I_2)MA = \begin{pmatrix} \delta_1^{-1} & \delta_1^{-1} & -\delta_1^{-1} & -\delta_1^{-1} \\ \delta_2^{-1} \left(\frac{d}{dt} + \eta_1 \right) & \delta_2^{-1} \left(\frac{d}{dt} - \eta_1 \right) & \delta_2^{-1} \eta_2 & -\delta_2^{-1} \eta_2 \end{pmatrix},$$

A right Smith-Jacobson decomposition of F , namely $VF\tilde{Q} = (I_2, 0_{2,2})$, is given by

$$\tilde{Q} = \begin{pmatrix} 1 & \frac{1}{2\eta_1} & \frac{1}{2\eta_1} \left(-\frac{d}{dt} + (\eta_1 - \eta_2) \right) & \frac{1}{2\eta_1} \left(-\frac{d}{dt} + (\eta_1 + \eta_2) \right) \\ 0 & -\frac{1}{2\eta_1} & \frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 + \eta_2) \right) & \frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 - \eta_2) \right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (40)$$

with $\bar{\pi} = \delta_1 \delta_2$, and where \tilde{Q} is obtained as the product of elementary actions \tilde{Q}_1 and \tilde{Q}_2 :

$$\tilde{Q}_1 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (41)$$

$$\tilde{Q}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2\eta_1} & \frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 + \eta_2) \right) & \frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 - \eta_2) \right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

thus showing that F is hyper-regular.

According to step 2 of the algorithm, we compute $Q = \tilde{Q} \begin{pmatrix} 0_{2,2} \\ I_2 \end{pmatrix}$ and $R = N(I_2, 0_{2,2})MAQ$:

$$Q = \begin{pmatrix} \frac{1}{2\eta_1} \left(-\frac{d}{dt} + (\eta_1 - \eta_2) \right) & \frac{1}{2\eta_1} \left(-\frac{d}{dt} + (\eta_1 + \eta_2) \right) \\ \frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 + \eta_2) \right) & \frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 - \eta_2) \right) \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (42)$$

$$R = \begin{pmatrix} R_{1,1} & R_{1,2} \\ \delta_2^2 & 1 \end{pmatrix}$$

with

$$\begin{aligned} R_{1,1} &= \frac{1}{2\eta_1} \left(-\frac{d}{dt} + (\eta_1 - \eta_2) \right) + \frac{\delta_1^2}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 + \eta_2) \right) \\ R_{1,2} &= \frac{1}{2\eta_1} \left(-\frac{d}{dt} + (\eta_1 + \eta_2) \right) + \frac{\delta_1^2}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 - \eta_2) \right) \end{aligned} \quad (43)$$

We indeed have $\pi_R = 1$.

Therefore, setting $x = Qy$ and $u = Ry$, we obtain the expressions

$$\begin{pmatrix} \psi_1(t) \\ \phi_1(t) \\ \psi_2(t) \\ \phi_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\eta_1} (-\dot{y}_1(t) + (\eta_1 - \eta_2)y_1(t) - \dot{y}_2(t) + (\eta_1 + \eta_2)y_2(t)) \\ \frac{1}{2\eta_1} (\dot{y}_1(t) + (\eta_1 + \eta_2)y_1(t) + \dot{y}_2(t) + (\eta_1 - \eta_2)y_2(t)) \\ y_1(t) \\ y_2(t) \end{pmatrix}. \quad (44)$$

and

$$\begin{aligned} u_1(t) &= \frac{1}{2\eta_1} [\dot{y}_1(t - \tau_1) - \dot{y}_1(t + \tau_1) + \dot{y}_2(t - \tau_1) - \dot{y}_2(t + \tau_1) \\ &\quad + (\eta_1 + \eta_2)(y_1(t - \tau_1) + y_2(t + \tau_1)) + (\eta_1 - \eta_2)(y_1(t + \tau_1) + y_2(t - \tau_1))] \\ u_2(t) &= y_1(t - \tau_2) + y_2(t + \tau_2). \end{aligned} \quad (45)$$

Further, according to step 3, we compute \tilde{P} and W of a Smith-Jacobson decomposition of Q , namely $\tilde{P}QW = \begin{pmatrix} I_2 \\ 0_{2,2} \end{pmatrix}$. We find $W = I_2$ and $\tilde{P} = \tilde{P}_4\tilde{P}_3\tilde{P}_2\tilde{P}_1$ with

$$\begin{aligned} \tilde{P}_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{P}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 + \eta_2) \right) & 1 & 0 & 0 \\ \frac{1}{2\eta_1} \left(\frac{d}{dt} - (\eta_1 - \eta_2) \right) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \tilde{P}_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{P}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2\eta_1} \left(\frac{d}{dt} - (\eta_1 + \eta_2) \right) & 1 & 0 \\ 0 & -\frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 - \eta_2) \right) & 0 & 1 \end{pmatrix}, \end{aligned}$$

thus

$$\tilde{P} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \frac{1}{2\eta_1} \left(\frac{d}{dt} - (\eta_1 - \eta_2) \right) & \frac{1}{2\eta_1} \left(\frac{d}{dt} - (\eta_1 + \eta_2) \right) \\ 0 & 1 & -\frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 + \eta_2) \right) & -\frac{1}{2\eta_1} \left(\frac{d}{dt} + (\eta_1 - \eta_2) \right) \end{pmatrix} \quad (46)$$

and

$$P = W \begin{pmatrix} I_2 & 0_{2,2} \end{pmatrix} \tilde{P} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (47)$$

Finally, setting $y = Px$, we get $y_1 = \psi_2$, $y_2 = \phi_2$ and $\pi_P = 1$.

Taking the least common multiple of $(1, 1, \delta_1\delta_2)$, we get $\pi = \delta_1\delta_2$. It is then immediately seen that $y_1 = \psi_2$ and $y_2 = \phi_2$ is a $\delta_1\delta_2$ -flat output.

Remark 8. For multiple delays $\delta_1, \delta_2, \dots, \delta_n$, a flat output for which the polynomial π is restricted to a monomial (i.e. of the form $\delta_1^{s_1}\delta_2^{s_2}\dots\delta_n^{s_n}$) is called δ -flat output in [31]. This is the case here with $n = 2$ and $s_1 = s_2 = 1$.

Remark 9. In [23], a different solution $y_1 = \delta_1\phi_1 - u$, $y_2 = \phi_1 + \psi_1$ has been proposed.

5. Concluding Remarks

In this paper, a direct characterization of π -flat outputs for linear time-varying, time-delay systems, with coefficients that are meromorphic functions of time is obtained, yielding a constructive algorithm for their computation. The proposed approach is based on the Smith-Jacobson decomposition of a polynomial matrix over the Principal Ideal Domain $\mathfrak{K}(\delta)[\frac{d}{dt}]$, containing the original ring of multivariate polynomials $\mathfrak{K}[\delta, \frac{d}{dt}]$. The fact that the computations are done in a larger ring, which is a principal ideal ring, makes them elementary and their localization at the powers of a δ -polynomial results from an easy calculation of least common multiple of a finite set of polynomials of δ . It is remarkable, however, that the π -flatness criterion only involves properties of the system matrices over the extended ring $\mathfrak{K}(\delta)[\frac{d}{dt}]$. Several examples are presented to illustrate the simplicity of the approach. Translating our algorithm in a computer algebra programme, e.g. in Maple or Mathematica, might be relatively easy and will be the subject of future works.

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Appendix

We recall from section 2.1 the following theorem:

Theorem 3 (Smith-Jacobson decomposition [5, 11]). *Let $M \in \mathcal{M}_{p,q}(\delta)[\frac{d}{dt}]$ be an arbitrary polynomial matrix of size $p \times q$. There exist unimodular matrices $U \in \mathcal{U}_p(\delta)[\frac{d}{dt}]$ and $V \in \mathcal{U}_q(\delta)[\frac{d}{dt}]$ such that:*

- $UMV = (\Delta_p | 0_{p,q-p})$ if $p \leq q$,
- $UMV = \begin{pmatrix} \Delta_q \\ 0_{p-q,q} \end{pmatrix}$ if $p > q$.

In both cases, $\Delta_p \in \mathcal{M}_{p,p}(\delta)[\frac{d}{dt}]$ and $\Delta_q \in \mathcal{M}_{q,q}(\delta)[\frac{d}{dt}]$ are diagonal matrices whose diagonal elements $(d_1, \dots, d_\sigma, 0, \dots, 0)$ are such that d_i is a nonzero $\frac{d}{dt}$ -polynomial for $i = 1, \dots, \sigma$, with coefficients in $\mathfrak{K}(\delta)$, and is a divisor of d_j for all $i \leq j \leq \sigma$.

A. Elementary Actions and Unimodular Matrices

The group of unimodular matrices admits a finite set of generators corresponding to the following *elementary right and left actions*:

- *right actions* consist in permuting two columns, right multiplying a column by a non zero function of $\mathfrak{K}(\delta)$, or adding the j th column right multiplied by an arbitrary element of $\mathfrak{K}(\delta)[\frac{d}{dt}]$ to the i th column, for arbitrary i and j ;
- *left actions* consist, symmetrically, in permuting two lines, left multiplying a line by a non zero function of $\mathfrak{K}(\delta)$, or adding the j th line left multiplied by an arbitrary element of $\mathfrak{K}(\delta)[\frac{d}{dt}]$ to the i th line, for arbitrary i and j .

Every elementary action may be represented by an *elementary unimodular matrix* of the form $T_{i,j}(p) = I_\nu + 1_{i,j}p$ with $1_{i,j}$ the matrix made of a single 1 at the intersection of line i and column j , $1 \leq i, j \leq \nu$, and zeros elsewhere, with p an arbitrary element of $\mathfrak{K}(\delta)[\frac{d}{dt}]$, and with $\nu = m$ for right actions and $\nu = n$ for left actions. One can easily prove that:

- right multiplication $MT_{i,j}(p)$ consists in adding the i th column of M right multiplied by p to the j th column of M , the remaining part of M being left unchanged,
- left multiplication $T_{i,j}(p)M$ consists in adding the j th line of M left multiplied by p to the i th line of M , the remaining part of M being left unchanged,
- $T_{i,j}^{-1}(p) = T_{i,j}(-p)$,
- $T_{i,j}(1)T_{j,i}(-1)T_{i,j}(1)M$ (resp. $MT_{i,j}(1)T_{j,i}(-1)T_{i,j}(1)$) is the permutation matrix replacing the j th line of M by the i th one and replacing the j th one of M by the i th one multiplied by -1 , all other lines remaining unchanged (resp. the permutation matrix replacing the i th column of M by the j th one multiplied by -1 and replacing the j th one by the i th one, all other columns remaining unchanged).

Every unimodular matrix V (left) and U (right) may be obtained as a product of such elementary unimodular matrices, possibly with a diagonal matrix $D(\alpha) = \text{diag}\{\alpha_1, \dots, \alpha_\nu\}$ with $\alpha_i \in \mathfrak{K}$, $\alpha_i \neq 0$, $i = 1, \dots, \nu$, at the end since $T_{i,j}(p)D(\alpha) = D(\alpha)T_{i,j}(\frac{1}{\alpha_i}p\alpha_j)$.

In addition, every unimodular matrix U is obtained by such a product: its decomposition yields $VU = I$ with V finite product of the $T_{i,j}(p)$'s and a diagonal matrix. Thus, since the inverse of any $T_{i,j}(p)$ is of the same form, namely $T_{i,j}(-p)$, and since the inverse of a diagonal matrix is diagonal, it results that $V^{-1} = U$ is a product of elementary matrices of the same form, which proves the assertion.

B. The Smith-Jacobson Decomposition Algorithm

The Smith-Jacobson decomposition algorithm of the matrix M consists first in permuting columns (resp. lines) to put the element of lowest degree in upper left position, denoted by $m_{1,1}$, or creating this element by euclidean division (in $\mathfrak{K}(\delta)[\frac{d}{dt}]$) of two or more elements of the first line (resp. column) by suitable right actions (resp. left action). Then right divide all the other elements $m_{1,k}$ (resp. left divide the $m_{k,1}$) of the new first line (resp. first column) by $m_{1,1}$. If one of the rests is non zero, say $r_{1,k}$ (resp. $r_{k,1}$), subtract the corresponding column (resp. line) to the first column (resp. line) right multiplied (resp. left) by the corresponding quotient $q_{1,k}$ defined by the right euclidean division $m_{1,k} = m_{1,1}q_{1,k} + r_{1,k}$ (resp. $q_{k,1}$ defined by $m_{k,1} = q_{k,1}m_{1,1} + r_{k,1}$). Then right multiplying all the columns by the corresponding quotients $q_{1,k}$, $k = 2, \dots, \nu$ (resp. left multiplying lines by $q_{k,1}$, $k = 2, \dots, \mu$), we iterate this process with the transformed first line (resp. first column) until it becomes $(m_{1,1}, 0, \dots, 0)$ (resp. $(m_{1,1}, 0, \dots, 0)^T$ where T means transposition). We then apply the same algorithm to the second line starting from $m_{2,2}$ and so on. To each transformation of lines and columns correspond a left or right elementary unimodular matrix and the unimodular matrix V (resp. U) is finally obtained as the product of all left (resp. right) elementary unimodular matrices so constructed.

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